1. Use the chain rule to find the indicated derivatives:

   (a) $g'(t)$, where

$$g(t) = f(x(t), y(t)) \qquad f(x,y) = x^2y - \sin(y)$$
$$x(t) = \sqrt{t^2+1} \qquad y(t) = e^t$$

Using the chain rule for functions of more than one variable, we know

$$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt},$$

so I need to find those four derivatives.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2xy = 2e^t\sqrt{t^2+1} \\
\frac{\partial f}{\partial y} &= x^2 - \cos(y) = t^2 + 1 - \cos(e^t)
\end{aligned}$$

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2}(t^2+1)^{-1/2}(2t) = \frac{t}{\sqrt{t^2+1}} \\
\frac{dy}{dt} &= e^t.
\end{aligned}$$

Putting all this together, we get

$$\begin{aligned}
g'(t) &= \left(2e^t\sqrt{t^2+1}\right)\left(\frac{t}{\sqrt{t^2+1}}\right) + \left(t^2+1-\cos(e^t)\right)\left(e^t\right) \\
&= 2te^t + (t^2+1-\cos(e^t))e^t \\
&= e^t(t^2+2t+1-\cos(e^t)) \\
&= e^t[(t+1)^2 - \cos(e^t)]
\end{aligned}$$

(b) $\dfrac{\partial g}{\partial u}$ and $\dfrac{\partial g}{\partial v}$, where

$$g(u,v) = f(x(u,v), y(u,v)) \qquad f(x,y) = 4x^2y^3$$
$$x(u,v) = u^3 - v\sin(u) \qquad y(u,v) = 4u^2$$

Because this time my two independent variables are each themselves functions of two variables, I'm going to have two partial derivatives, each using the chain rule. Remember to use tree diagrams to help you come up with the right formulae. Unfortunately, I can't easily draw them here.

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \qquad \frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

I will again calculate each piece, and do the appropriate substitution, before putting everything together:

$$\frac{\partial f}{\partial x} = 8xy^3 = 8(u^3 - v\sin(u))(4u^2)^3 = 512u^6(u^3 - v\sin(u))$$
$$\frac{\partial f}{\partial y} = 12x^2y^2 = 12(u^3 - v\sin(u))^2(4u^2)^2 = 192u^4(u^3 - v\sin(u))^2$$

$$\frac{\partial x}{\partial u} = 3u^2 - v\cos(u)$$
$$\frac{\partial x}{\partial v} = -\sin(u)$$

$$\frac{\partial y}{\partial u} = 8u$$
$$\frac{\partial y}{\partial v} = 0$$

Putting it all together, we get

$$\begin{aligned}
\frac{\partial g}{\partial u} &= 512u^6(u^3 - v\sin(u))(3u^2 - v\cos(u)) + 192u^4(u^3 - v\sin(u))^2(8u) \\
&= 512u^6(u^3 - v\sin(u)(3u^2 - v\cos(u)) + 1536u^5(u^3 - v\sin(u)) \\
&= 512u^5(u^3 - v\sin(u))(3u^3 - uv\cos(u) + 3)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial g}{\partial v} &= 512u^6(u^3 - v\sin(u))(-\sin(u)) + 192u^4(u^3 - v\sin(u))^2(0) \\
&= -512u^6\sin(u)(u^3 - v\sin(u))
\end{aligned}$$

2. By drawing tree diagrams, figure out the chain rules for the following general composite functions:

   (a) $g(t) = f(x(t), y(t), z(t), w(t))$

   I can't easily draw the tree diagram, but here's the final result:

$$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial w}\frac{dw}{dt}$$

   (b) $g(u, v) = f(x(r, s), y(r, s, t))$, where $r$, $s$, and $t$ are all functions of $u$ and $v$.

   And again, here's the final result:

$$\begin{aligned}
\frac{\partial g}{\partial u} &= \frac{\partial f}{\partial x}(\frac{\partial x}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial x}{\partial s}\frac{\partial s}{\partial u}) + \frac{\partial f}{\partial y}(\frac{\partial y}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial y}{\partial s}\frac{\partial s}{\partial u} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial u}) \\
\frac{\partial g}{\partial v} &= \frac{\partial f}{\partial x}(\frac{\partial x}{\partial r}\frac{\partial r}{\partial v} + \frac{\partial x}{\partial s}\frac{\partial s}{\partial v}) + \frac{\partial f}{\partial y}(\frac{\partial y}{\partial r}\frac{\partial r}{\partial v} + \frac{\partial y}{\partial s}\frac{\partial s}{\partial v} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial v})
\end{aligned}$$

3. Use the chain rule twice to find $g''(t)$ if $g(t) = f(x(t), y(t))$.

As always, to find the second derivative, one must first find the first derivative:

$$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

To find the second derivative, we need to differentiate this with respect to $t$ again. I can tell this is going to involve some thought, but I'll just do it one step at a time. First off, I see that I have a sum, so I'll break that into two pieces first:

$$g''(t) = \frac{d}{dt}\left(\frac{\partial f}{\partial x}\frac{dx}{dt}\right) + \frac{d}{dt}\left(\frac{\partial f}{\partial y}\frac{dy}{dt}\right).$$

Looking at each piece of the sum, I can see that I need to differentiate some products. So far, I haven't had to think about the chain rule yet, so I'm just going to plow ahead. Using the product rule on both products in the sum:

$$g''(t) = \left[\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{dx}{dt} + \frac{\partial f}{\partial x}\cdot\frac{d}{dt}\left(\frac{dx}{dt}\right)\right] + \left[\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{dy}{dt} + \frac{\partial f}{\partial y}\cdot\frac{d}{dt}\left(\frac{dy}{dt}\right)\right].$$

Now I'm beginning to see where the chain rule is going to come in – both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are really functions of $x(t)$ and $y(t)$, so I'll need to use the chain rule on them. Before I get to those, however, I can at least do something about the other two terms.

$$g''(t) = \left[\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{dx}{dt} + \frac{\partial f}{\partial x}\cdot\frac{d^2x}{dt^2}\right] + \left[\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{dy}{dt} + \frac{\partial f}{\partial y}\cdot\frac{d^2y}{dt^2}\right].$$

Repeating the result from the previous page, we have

$$g''(t) = \left[\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{dx}{dt} + \frac{\partial f}{\partial x}\cdot\frac{d^2x}{dt^2}\right] + \left[\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{dy}{dt} + \frac{\partial f}{\partial y}\cdot\frac{d^2y}{dt^2}\right].$$

Now I'm ready to apply the chain rule to both $\frac{\partial f}{\partial x}(x(t), y(t))$ and $\frac{\partial f}{\partial y}(x(t), y(t))$.

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{dy}{dt} \\
&= \frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dt} \\
\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt} \\
&= \frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}
\end{aligned}$$

Putting it all together, I get:

$$\begin{aligned}
g''(t) &= \left\{\left[\frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dt}\right]\cdot\frac{dx}{dt} + \frac{\partial f}{\partial x}\cdot\frac{d^2x}{dt^2}\right\} \\
&\qquad + \left\{\left[\frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}\right]\frac{dy}{dt} + \frac{\partial f}{\partial y}\cdot\frac{d^2y}{dt^2}\right\} \\
&= \frac{\partial^2 f}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + \frac{\partial^2 f}{\partial y \partial x}\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} \\
&\qquad + \frac{\partial^2 f}{\partial x \partial y}\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2}
\end{aligned}$$