Let $g(t) = f(x(t), y(t))$.

Recall: If both f_x and f_y are continuous, then f is differentiable.

$$
\begin{aligned}
g'(t) &= \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t), y(t+\Delta t)) - f(t)}{\Delta t} \\
g'(t) &= \lim_{\Delta t \to 0} \frac{[f(x(t+\Delta t), y(t+\Delta t)) - \mathbf{f(x(t), y(t+\Delta t))}]}{\Delta t} \\
&\quad + \lim_{\Delta t \to 0} \frac{[\mathbf{f(x(t), y(t+\Delta t))} - f(x(t), y(t))]}{\Delta t}
\end{aligned}
$$

Let

$$
\begin{aligned}
\Delta x &= x(t+\Delta t) - x(t), \text{ so } x(t+\Delta t) = x + \Delta x \\
\Delta y &= y(t+\Delta t) - y(t) \text{ so } y(t+\Delta t) = y + \Delta y
\end{aligned}
$$

Then

$$
\begin{aligned}
g'(t) &= \lim_{\Delta t \to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x} \cdot \frac{\Delta x}{\Delta t} \\
&\quad + \lim_{\Delta t \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} \cdot \frac{\Delta y}{\Delta t}
\end{aligned}
$$

Since both x and y are differentiable, as $\Delta t \to 0$, so do Δx and Δy. We therefore end up with

$$
\boxed{[g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}
$$

1. Use the chain rule to find the indicated derivatives:

   (a) $g'(t)$, where

$$g(t) = f(x(t), y(t)) \qquad f(x, y) = x^2 y - \sin(y)$$
$$x(t) = \sqrt{t^2 + 1} \qquad y(t) = e^t$$

   (b) $\dfrac{\partial g}{\partial u}$ and $\dfrac{\partial g}{\partial v}$, where

$$g(u, v) = f(x(u, v), y(u, v)) \qquad f(x, y) = 4x^2 y^3$$
$$x(u, v) = u^3 - v\sin(u) \qquad y(u, v) = 4u^2$$

2. By drawing tree diagrams, figure out the chain rules for the following general composite functions:

   (a) $g(t) = f(x(t), y(t), z(t), w(t))$

   (b) $g(u, v) = f(x(r, s), y(r, s, t))$, where r, s, and t are all functions of u and v.

3. Use the chain rule twice to find $g''(t)$ if $g(t) = f(x(t), y(t))$.