For the vector-valued function $\vec{\mathbf{r}}(t) =< 4\cos(\pi t), 4\sin(\pi t), t >$, find

1. the unit tangent vector $\vec{\mathbf{T}}(t)$ at $t = 0$

$$\begin{aligned} \vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}'}(t)}{\|\vec{\mathbf{r}'}(t)\|} \\ &= \frac{< -4\pi\sin(\pi t), 4\pi\cos(\pi t), 1 >}{\sqrt{16\pi^2+1}} \\ \vec{\mathbf{T}}(0) &= \frac{< 0, 4\pi, 1 >}{\sqrt{16\pi^2+1}} \end{aligned}$$

2. the principal unit normal vector $\vec{\mathbf{N}}(t)$ at $t = 0$

$$\begin{aligned} \vec{\mathbf{N}}(t) &= \frac{\frac{1}{\sqrt{16\pi^2+1}} < -4\pi^2\cos(\pi t), -4\pi^2\sin(\pi t), 0 >}{\frac{1}{\sqrt{16\pi^2+1}}(\sqrt{16\pi^4})} \\ &= \frac{1}{4\pi^2} < -4\pi^2\cos(\pi t), -4\pi^2\sin(\pi t), 0 > \\ &= < -\cos(\pi t), -\sin(\pi t), 0 > \\ \vec{\mathbf{N}}(0) &= < -1, 0, 0 > \end{aligned}$$

3. Check to make sure $\vec{\mathbf{T}}(0)$ and $\vec{\mathbf{N}}(0)$ are orthogonal and have length 1.

$$\frac{< 0, 4\pi, 1 >}{\sqrt{16\pi^2+1}} \cdot < -1, 0, 0 >= 0 + 0 + 0 = 0.$$

   Both $\vec{\mathbf{T}}(0)$ and $\vec{\mathbf{N}}(0)$ obviously have length 1.

4. Find the tangential and normal components of acceleration.

   - To find a_T:

– Using that $a_T = \dfrac{d^2 s}{dt^2} = \dfrac{d}{dt}\left(\|\vec{\mathbf{r}}\,'(t)\|\right)$:

$$\begin{aligned}
a_T &= \frac{d^2 s}{dt^2} \\
&= \frac{d}{dt}\left(\|\vec{\mathbf{r}}\,'(t)\|\right) \\
&= \frac{d}{dt}\left(\| < -4\pi\sin(\pi t), 4\pi\cos(\pi t), 1 > \|\right) \\
&= \frac{d}{dt}\sqrt{16\pi^2\sin^2(\pi t) + 16\pi^2\cos^2(\pi t) + 1} \\
&= \frac{d}{dt}\sqrt{16\pi^2 + 1} \\
&= 0
\end{aligned}$$

– Using that a_T is the component of $\vec{\mathbf{a}}(t)$ in the direction of $\vec{\mathbf{T}}(t)$:

$$\begin{aligned}
a_T &= \text{comp}_{\vec{\mathbf{T}}(t)}\vec{\mathbf{a}}(t) \\
&= \vec{\mathbf{a}}(t)\cdot\frac{\vec{\mathbf{T}}(t)}{\|\vec{\mathbf{T}}(t)\|} = \vec{\mathbf{a}}(t)\cdot\vec{\mathbf{T}}(t) \\
&= \vec{\mathbf{r}}\,''(t)\cdot\frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} \\
&= \frac{1}{\sqrt{16\pi^2+1}} < -16\pi^2\cos(\pi t), -16\pi^2\sin(\pi t), 0 > \\
&\qquad \cdot < -4\pi\sin(\pi t), 4\pi\cos(\pi t), 1 > \\
&= \frac{1}{\sqrt{16\pi^2+1}}(64\pi^3\sin(\pi t)\cos(\pi t) - 64\pi^3\sin(\pi t)\cos(\pi t) + 0) \\
&= 0
\end{aligned}$$

Whether we used the original definition of a_T, or whether we used the concept of components and projections, we get that the component of the acceleration in the tangent vector's direction is 0. Does this make sense?

Since the acceleration is orthogonal to the tangent vector (which we now know, because we got that $\vec{\mathbf{a}}(t)\cdot\vec{\mathbf{T}}(t) = 0$), we wouldn't expect the acceleration to have any component in the direction of the tangent vector. It's all in the direction of the normal vector!

This makes find a_N easier.

- To find a_N:

$$\begin{aligned} \vec{\mathbf{a}}(t) &= a_T \vec{\mathbf{T}}(t) + a_N \vec{\mathbf{N}}(t) \\ < -16\pi^2 \cos(\pi t), -16\pi^2 \sin(\pi t), 0 > &= 0 + a_N < -\cos(\pi t), -\sin(\pi t), 0 > \\ \Rightarrow a_N &= 16\pi^2 \end{aligned}$$