Find the following integrals on this page and on the next, and *check your answers by differentiating your results!!*

1. $\displaystyle\int \frac{1}{\sqrt{1-x}}\,dx \qquad (u = 1-x)$

   Let $u = 1-x$. Then $du = -dx$.
   Substituting back into the integral, we find

$$\int \frac{1}{\sqrt{1-x}}\,dx = -\int \frac{1}{\sqrt{u}}\,du = -\int u^{-1/2}\,du = -2u^{1/2} + C = -2\sqrt{1-x} + C$$

2. $\displaystyle\int x\sin(\pi x^2)\,dx \qquad (u = \pi x^2)$

   Let $u = \pi x^2$. Then $du = 2\pi x\,dx$, so $\dfrac{1}{2\pi}\,du = x\,dx$.
   Substituting back in to the integral, we find

$$\int x\sin(\pi x^2)\,dx = \frac{1}{2\pi}\int \sin(u)\,du = \frac{1}{2\pi} * (-\cos(u)) + C = -\frac{1}{2\pi}\cos(\pi x^2) + C$$

3. $\displaystyle\int \frac{x}{1+x^2}\,dx \qquad (u = 1+x^2)$

   Let $u = 1+x^2$. Then $du = 2x\,dx$, so $\dfrac{1}{2}\,du = x\,dx$.
   Substituting back in to the integral, we find

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{\ln(u)}{2} + c = \frac{\ln(1+x^2)}{2} + C$$

4. $\displaystyle\int \frac{x}{1+x^4}\,dx \qquad (u = x^2)$

   Let $u = x^2$. Then $du = 2x\,dx$, so $\dfrac{1}{2}\,du = x\,dx$.
   Substituting back in to the integral, we find

$$\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{\arctan(u)}{2} + C = \frac{\arctan(x^2)}{2} + C$$

5. $\int x\cos(3+x^2)\,dx$

Let $u = 3+x^2$. Then $du = 2x\,dx$, so $\frac{1}{2}\,du = x\,dx$.
Substituting back in to the integral, we find

$$\int x\cos(3+x^2)\,dx = \frac{1}{2}\int \cos(u)\,du = \frac{1}{2}\sin(u)+C = \frac{1}{2}\sin(3+x^2)+C$$

6. $\int \sin(x)e^{\cos(x)}\,dx$

Let $u = \cos(x)$. Then $du = -\sin(x)\,dx$, so $-du = \sin(x)\,dx$.
Substituting back in to the integral, we find

$$\int \sin(x)e^{\cos(x)}\,dx = -\int e^u\,du = -e^u + C = -e^{\cos(x)} + C$$

7. $\int \frac{\sqrt{5-\ln(x)}}{x}\,dx$

Let $u = 5-\ln(x)$. Then $du = -\frac{1}{x}\,dx$, so $-du = \frac{1}{x}\,dx$.
Then

$$\int \frac{\sqrt{5-\ln(x)}}{x}\,dx = -\int \sqrt{u}\,du = -\int u^{1/2}\,du = -\frac{2}{3}u^{3/2}+C = \frac{-2(5-\ln(x))^{3/2}}{3}+C$$

8. $\int \frac{x}{\sqrt{1-x^4}}\,dx$

Let $u = x^2$. Then $du = 2x\,dx$, so $\frac{1}{2}\,du = x\,dx$.
Then

$$\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{2}\arcsin(u)+C = \frac{1}{2}\arcsin(x^2)+C$$

9. $\int \frac{1}{x\ln(x)}\,dx$

Let $u = \ln(x)$. Then $du = \frac{1}{x}\,dx$.
Then

$$\int \frac{1}{x\ln(x)}\,dx = \int \frac{1}{u}\,du = \ln(u)+C = \ln(\ln(x))+C$$

Find the derivative of arctan$(x)$ by following steps similar to those we used to find the derivative of arcsin$(x)$.

Let $y = \arctan(x)$. We want to find $\frac{dy}{dx}$.

Because arctan$(x)$ and tan$(x)$ are inverse functions,

$$y = \arctan(x) \iff x = \tan(y)$$

We of course know how to differentiate tan$(y)$. Remember we want to differentiate with respect to $x$. Since $y$ is a function of $x$, we'll need to use the chain rule when we differentiate tan$(y)$. (This is called implicit differentiation, by the way).

Differentiating both sides of $x = \tan(y)$ with respect to $x$, we get

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan(y)) \Rightarrow 1 = \sec^2 y \frac{dy}{dx}.$$

Since $\frac{dy}{dx}$ is what we are looking for, we'll solve the above equation for it by dividing both sides by $\sec^2(y)$.

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \cos^2(y).$$

We're nearly done, but we can not say that the derivative of $y$ with respect to $x$ is a function of $y$. That's a circular answer. So we need to find $\sec^2(y)$ in terms of $x$.

$$\frac{dy}{dx} = \frac{1}{\sec^2(\arctan(x))} = \cos^2(\arctan(y)).$$

There are two ways to simlify this: use trigonometric identities, or draw right triangles where the second angle is $y$.

If I'm going to use trig identities, then what I'm going to be looking for is a a way to rewrite one of the above two expressions for $\frac{dy}{dx}$ in terms of tan(arctan$(x)$), because I know that this is just $x$.

$$\begin{aligned} \cos^2(x) + \sin^2(x) &= 1 \\ \text{Therefore, } 1 + \tan^2(x) &= \sec^2(x) \end{aligned}$$

Thus

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2(\arctan(x))} = \frac{1}{1 + x^2}.$$