

1. Find the maximum and minimum value of $f(x) = x^3 - 3x + 5$ over the interval $[0, 2]$.

In general, the maximum and minimum values of a function f can only *occur* at the following places:

- stationary points of f (i.e. where $f'(x) = 0$)
- places where the derivative does not exist
- at the endpoints of the interval.

Be aware: This doesn't mean each of these points *is* some sort of maximum or minimum. It just means that instead of testing every single x -value between $x = 0$ and $x = 2$, we can narrow the field down to points of the above three types.

To find the stationary points and places where the derivative does not exist, we find the derivative:

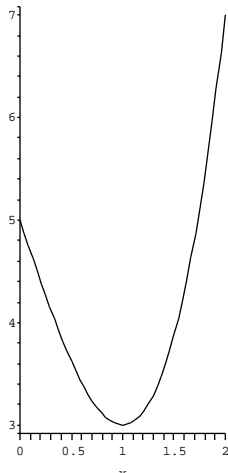
$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1) \quad f'(x) = 0 \text{ at } x = 1, x = -1.$$

Since $x = -1$ isn't in the interval $[0, 2]$, the only places f can attain its maximum and minimum value are therefore $x = 0$, $x = 1$, and $x = 2$.

x	$f(x)$
0	5
1	3
2	7

Therefore, without even looking at the graph of $f(x)$, I know that the highest point of f between 0 and 2 occurs at $x = 2$ and the lowest occurs at $x = 1$.

To check this, I can of course look at the graph.



Sure enough, the highest point occurs not at a flat part (local max) but at the endpoint $x = 2$, while the lowest point occurs at the local minimum at $x = 1$.

2. Find a function with stationary points at $x = -1$, $x = 0$ and $x = 2$. Check that your function behaves properly by graphing it.

Stationary points are where the derivative is 0. So we want to construct a function f that has $f'(-1) = 0$, $f'(0) = 0$, and $f'(2) = 0$. How can I create such a function? There are probably an infinite number of such functions, but here's what I think is the easiest way:

What we're going to do is create f' first, and then get f from there.

- Could $f'(x) = 0$ —that is, could $f'(x)$ be the horizontal line that is the x -axis? Well, if $f'(x) = 0$, then $f'(-1) = 0$, $f'(0) = 0$ and $f'(2) = 0$, but these aren't the *only* points where f' is 0, so we wouldn't have created a function with only these stationary points.
- The next-easiest function f' that has $f'(0) = 0$ is $f'(x) = x$. Unfortunately, if $f'(x) = x$, then $f'(-1) \neq 0$ and $f'(2) \neq 0$. But we can build on this idea!

- Along the same ideas as the previous thought, $f'(x) = x + 1$ has $f'(-1) = 0$, but $f'(0) \neq 0$. *However* (and here's the big brilliant idea!), the function $x(x + 1)$ is 0 at both $x = 0$ and $x = -1$. Building upon that idea even farther: since $x - 2$ is 0 at $x = 2$, if we throw that factor into the mix as well, what do we get?

$$f'(x) = x(x+1)(x-2) \quad f'(0) = 0 \quad f'(-1) = 0 \quad f'(2) = 0.$$

Thus $f'(x) = x(x + 1)(x - 2)$ will work as our derivative.

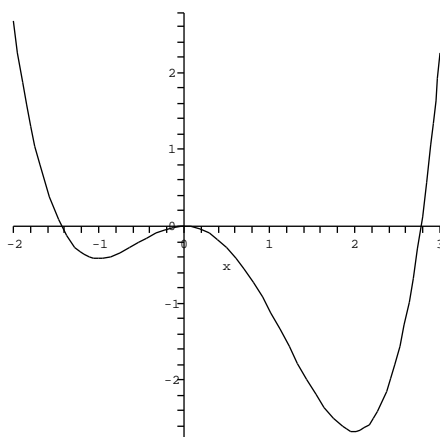
In that case, what is $f(x)$? We'll have to antidifferentiate $f'(x)$. In order to do that, we first must simplify $f'(x)$:

$$f'(x) = x(x+1)(x-2) = x[(x+1)(x-2)] = x(x^2 - 2x + x - 2) = x(x^2 - x - 2) = x^3 - x^2 - 2x.$$

Antidifferentiating this, we find that one function with stationary points at $x = -1$, $x = 0$, and $x = 2$ would be

$$f(x) = \frac{x^4}{4} - \frac{x^3}{3} - x^2.$$

We can always check our answer, either by differentiating it and making sure we get stationary points at $x = 0$, $x = -1$ and $x = 2$ or simply by graphing our $f(x)$:



Sure enough, this has stationary points at $x = -1$, $x = 0$ and $x = 2$ just as desired.

3. For which values of k , if any, does the function $f(x) = (8x+k)/x^2$ have a local minimum at $x = 4$?

$f(x)$ will have a local minimum at $x = 4$ if

- $f'(4) = 0$, and
 - $f''(4) > 0$ (since then the function would be concave up there).
- First, let's see what I can tell about k just from knowing $f'(4) = 0$ – that is, just from knowing that f has a stationary point at $x = 4$.

I need to $f'(4)$ to be 0. So I need to find $f'(x)$, plug 4 in, set the result equal to 0, and see what that tells me about k . But right now, $f(x)$ isn't in a form we know how to differentiate, so I need to simplify/rewrite f first.

$$\begin{aligned} f(x) &= \frac{8x+k}{x^2} \\ &= \frac{8x}{x^2} + \frac{k}{x^2} \\ &= \frac{8}{x} + \frac{k}{x^2} \text{ if } x \neq 0 \\ &= 8x^{-1} + kx^{-2} \end{aligned}$$

Now f is in a form I can differentiate.

$$\begin{aligned} f'(x) &= -8x^{-2} - 2kx^{-3} \\ &= -\frac{8}{x^2} - \frac{2k}{x^3} \end{aligned}$$

I know that in order for $x = 4$ to be a local minimum, I need $f'(4) = 0$. Let's see where that takes us:

$$\begin{aligned} f'(4) &= -\frac{8}{16} - \frac{2k}{64} \\ &= -\frac{1}{2} - \frac{k}{32} \end{aligned}$$

Since $f'(4) = 0$, this means that

$$-\frac{1}{2} - \frac{k}{32} = 0 \Rightarrow \frac{k}{32} = -\frac{1}{2} \Rightarrow k = -16.$$

Conclusion: In order for f to have any sort of stationary point at $x = 4$, k must be -16 . But we don't know yet whether f has a local minimum, local maximum, or inflection point at $x = 4$ when $k = -16$, so even though it feels as if we're done, we're not!

- If we let $k = -16$, is $f''(4) > 0$?

With $k = -16$, we have that

$$\begin{aligned} f(x) &= (8x - 16)/x^2 \\ &= 8x^{-1} - 16x^{-2} \\ \Rightarrow f'(x) &= -8x^{-2} + 32x^{-3} \\ \Rightarrow f''(x) &= -16x^{-3} - 96x^{-4} \\ \Rightarrow f''(4) &= \frac{16}{4^3} - \frac{96}{4^4} \\ &= \frac{1}{4} - \frac{2^5 \cdot 3}{2^8} \\ &= \frac{1}{4} - \frac{3}{8} \\ &< 0 \end{aligned}$$

Because $f''(4)$ is negative at $x = 4$, f is concave down at $x = 4$, which means that f has a *maximum* at $x = 4$, not a *minimum* after all.

Conclusion: There is no value of k for which $x = 4$ is a local minimum of $f(x)$, although $k = -16$ does lead to $x = 4$ being a local maximum.

4. A particle moves along a straight line so that its velocity after t minutes is given by $v(t) = t^2$. How far does the particle travel between $t = 1$ and $t = 3$?

If its velocity is given by $v(t) = t^2$, then its position must be given $p(t) = t^3/3$. Thus at $t = 1$, it was at position $p(1) = 1/3$ and at $t = 3$ it was at position $p(3) = 27/3 = 9$. At no time in that interval did it turn around, since the velocity was always positive, and so the difference in position is the same as the distance traveled.

Thus it traveled $26/3$ units in those 2 minutes.