

Find the derivatives of the following functions.

Remember that you can verify your answers by graphing.

1. $f(x) = \ln(x^2 - x)$

Chain rule: $u = x^2 - x$ and $f(u) = \ln(u)$.

$$f'(x) = f'(u)u' = \frac{1}{x^2 - x} \cdot (2x - 1).$$

2. $f(x) = e^x \tan(x)$

Product rule: $u = e^x$ and $v = \tan(x)$.

$$f'(x) = uv' + u'v = e^x \sec^2(x) + e^x \tan(x).$$

3. $f(x) = \frac{\sec(x) + x^2}{\ln(x)}$

Quotient rule: $u = \sec(x) + x^2$, $v = \ln(x)$

$$f'(x) = \frac{vu' - uv'}{v^2} = \frac{\ln(x)(\sec(x)\tan(x) + 2x) - (\sec(x) + x^2)\left(\frac{1}{x}\right)}{(\ln(x))^2}$$

4. $f(x) = (3x^2 - 7)\sin(x)$

Product rule: $u = 3x^2 - 7$, $v = \sin(x)$

$$f'(x) = uv' + u'v = (3x^2 - 7)\cos(x) + (6x)\sin(x)$$

5. $f(x) = (\cot(x))^5 + 7$

Chain rule: - for $(\cot(x))^5$, let $u = \cot(x)$, $g(u) = u^5$.

$$f'(x) = g'(u)u' + (7)' = 5(\cot(x))^4 \cdot -\csc^2(x) + 0 = -5\cot^4(x)\csc^2(x)$$

6. $f(x) = \csc(x^4)$

Chain rule: $u = x^4$, $f(u) = \csc(u)$

$$f'(x) = f'(u)u' = -\csc(u)\cot(u) \cdot 4x^3 = -4x^3\csc(x^4)\cot(x^4)$$

7. $f(x) = 3^{2x \tan(x)}$

Chain rule (to start): $u = 2x \tan(x)$, $f(u) = 3^u$

$$f'(x) = f'(u)u' = \ln(3)3^{2x \tan(x)} \frac{d}{dx}(2x \tan(x))$$

Now, product rule: $u = 2x$, $v = \tan(x)$

$$f'(x) = \ln(3)3^{2x \tan(x)}(uv' + u'v) = \ln(3)3^{2x \tan(x)}(2x \sec^2 x + 2 \tan(x))$$

8. $f(x) = \frac{\sec(x^2)}{e^x}$

Quotient rule (to start): $u = \sec(x^2)$, $v = e^x$

$$f'(x) = \frac{vu' - uv'}{v^2} = \frac{e^x \frac{d}{dx}(\sec(x^2)) - \sec(x^2)e^x}{(e^x)^2}$$

Now, chain rule: $u = x^2$, $g(u) = \sec(u)$

$$\begin{aligned} f'(x) &= \frac{e^x(g'(u)u') - e^x \sec(x^2)}{e^{2x}} = \frac{e^x \sec(u) \tan(u)(2x) - e^x \sec(x^2)}{e^{2x}} \\ &= \frac{2xe^x \sec(x^2) \tan(x^2) - e^x \sec(x^2)}{e^{2x}} \end{aligned}$$

9. $f(x) = x^5 e^x \sin(x^3)$

Product rule (triple product) to start with: $u = x^5$, $v = e^x$,
 $w = \sin(x^3)$

$$f'(x) = uvw' + uv'w + u'vw = x^5 e^x \frac{d}{dx}(\sin(x^3)) + x^5 e^x \sin(x^3) + 5x^4 e^x \sin(x^3)$$

Now, chain rule: $u = x^3$, $g(u) = \sin(u)$

$$\begin{aligned} f'(x) &= x^5 e^x g'(u)u' + x^5 e^x \sin(x^3) + 5x^4 e^x \sin(x^3) \\ &= x^5 e^x \cos(x^3)3x^2 + x^5 e^x \sin(x^3) + 5x^4 e^x \sin(x^3) \end{aligned}$$

10. $f(x) = \sqrt{x^2 \cos(x)}$

Chain rule (to start): $u = x^2 \cos(x)$, $f(u) = \sqrt{u} = u^{1/2}$

$$f'(x) = f'(u)u' = \frac{1}{2}(x^2 \cos(x))^{-1/2} \frac{d}{dx}(x^2 \cos(x)) = \frac{1}{2\sqrt{x^2 \cos(x)}} \frac{d}{dx}(x^2 \cos(x))$$

Now, product rule: $u = x^2$, $v = \cos(x)$

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x^2 \cos(x)}}(uv' + u'v) = \frac{1}{2\sqrt{x^2 \cos(x)}}(x^2(-\sin(x)) + 2x \cos(x)) \\ &= \frac{-x^2 \sin(x) + 2x \cos(x)}{2\sqrt{x^2 \cos(x)}} \end{aligned}$$

11. $f(x) = \cos(\sin(x^5))$

Chain rule (to start): $u = \sin(x^5)$, $f(u) = \cos(u)$

$$f'(x) = f'(u)u' = -\sin(\sin(x^5)) \frac{d}{dx}(\sin(x^5))$$

Now, chain rule again! $u = x^5$, $g(u) = \sin(u)$

$$\begin{aligned} f'(x) &= -\sin(\sin(x^5))g'(u)u' = -\sin(\sin(x^5))\cos(x^5) \cdot 5x^4 \\ &= -5x^4 \sin(\sin(x^5))\cos(x^5) \end{aligned}$$

12. $f(x) = \left(\frac{x^2}{\ln(x)}\right)^3$

Chain rule, to start: $u = \frac{x^2}{\ln(x)}$, $f(u) = u^3$

$$f'(x) = f'(u)u' = 3\left(\frac{x^2}{\ln(x)}\right)^2 \cdot \frac{d}{dx}\left(\frac{x^2}{\ln(x)}\right)$$

Now, quotient rule! $u = x^2$, $v = \ln(x)$

$$\begin{aligned} f'(x) &= 3\left(\frac{x^2}{\ln(x)}\right)^2 \cdot \left(\frac{vu' - uv'}{v^2}\right) = 3\left(\frac{x^2}{\ln(x)}\right)^2 \cdot \left(\frac{\ln(x) \cdot 2x - x^2 \cdot \frac{1}{x}}{(\ln(x))^2}\right) \\ &= \frac{3x^4(2x \ln(x) - x)}{(\ln(x))^4} \end{aligned}$$

13. $f(x) = \log_4\left(e^{5x^2} \cos(e^x)\right) + \cot(e)$

Chain rule, to start with – on the first term: $u = e^{5x^2} \cos(e^x)$,
 $g(u) = \log_4(u)$.

Note: $\cot(e)$ is a constant, so its derivative is 0.

$$f'(x) = g'(u)u' = \frac{1}{\ln(4)} \cdot \frac{1}{u} \cdot \frac{d}{dx}(e^{5x^2} \cos(e^x)) + 0 = \frac{1}{\ln(4)e^{5x^2} \cos(e^x)} \frac{d}{dx}(e^{5x^2} \cos(e^x))$$

Next, product rule! $u = e^{5x^2}$, $v = \cos(e^x)$

$$\begin{aligned} f'(x) &= \frac{1}{\ln(4)e^{5x^2} \cos(e^x)} (uv' + u'v) \\ &= \frac{1}{\ln(4)e^{5x^2} \cos(e^x)} \left(e^{5x^2} \frac{d}{dx}(\cos(e^x)) + \frac{d}{dx}(e^{5x^2}) \cos(e^x) \right) \end{aligned}$$

And now, chain rule again: on two functions.

Let $u = e^x$, $g(u) = \cos(u)$, and let $v = 5x^2$, $h(v) = e^v$.

$$\begin{aligned} f'(x) &= \frac{1}{\ln(4)e^{5x^2} \cos(e^x)} \left(e^{5x^2} g'(u)u' + h'(v)v' \cos(e^x) \right) \\ &= \frac{1}{\ln(4)e^{5x^2} \cos(e^x)} \left(e^{5x^2} \cdot -\sin(e^x) \cdot e^x + e^{5x^2} \cdot 10x \cdot \cos(e^x) \right) \\ &= \frac{-e^{5x^2+x} \sin(e^x) + 10xe^{5x^2} \cos(e^x)}{\ln(4)e^{5x^2} \cos(e^x)} \end{aligned}$$

14. $f(x) = \frac{x^{42} + 3 \cos(4x^3 + \pi)}{e^{2x^2} - \sqrt{x}}$

Quotient rule, to start: $u = x^{42} + 3 \cos(4x^3 + \pi)$, $v = e^{2x^2} - \sqrt{x}$

$$\begin{aligned}
 f'(x) &= \frac{vu' - uv'}{v^2} = \frac{vu'}{v^2} - \frac{uv'}{v^2} \text{ I'm just writing it this way due to space considerations} \\
 &= \frac{(e^{2x^2} - \sqrt{x})(42x^{41} + \frac{d}{dx}(3 \cos(4x^3 + \pi)))}{(e^{2x^2} - \sqrt{x})^2} \\
 &\quad - \frac{(x^{42} + 3 \cos(4x^3 + \pi))(\frac{d}{dx}(e^{2x^2}) - \frac{1}{2}x^{-1/2})}{(e^{2x^2} - \sqrt{x})^2}
 \end{aligned}$$

Next, chain rule: on two functions:

Let $u = 4x^3 + \pi$, $g(u) = 3 \cos(u)$, and let $v = 2x^2$, $h(v) = e^v$.

$$\begin{aligned}
 f'(x) &= \frac{(e^{2x^2} - \sqrt{x})(42x^{41} + g'(u)u')}{(e^{2x^2} - \sqrt{x})^2} \\
 &\quad - \frac{(x^{42} + 3 \cos(4x^3 + \pi))(h'(v)v' - \frac{1}{2}x^{-1/2})}{(e^{2x^2} - \sqrt{x})^2} \\
 &= \frac{(e^{2x^2} - \sqrt{x})(42x^{41} + -3 \sin(4x^3 + \pi) \cdot 12x^2)}{(e^{2x^2} - \sqrt{x})^2} \\
 &\quad - \frac{(x^{42} + 3 \cos(4x^3 + \pi))(e^{2x^2} \cdot 4x - \frac{1}{2}x^{-1/2})}{(e^{2x^2} - \sqrt{x})^2}
 \end{aligned}$$