Find the derivatives of the following functions.
Remember that you can verify your answers by graphing.

1. $f(x) = \sec(x)$ Remember that $\sec(x) = \dfrac{1}{\cos(x)}$.

   First of all, I notice that I can rewrite $f(x)$ some more, to avoid the quotient rule.

$$f(x) = (\cos(x))^{-1}.$$

   Now I've got one function, $u(x) = \cos(x)$, inside another function $g(u) = u^{-1}$. Using the chain rule, I find

$$\begin{aligned} f'(x) &= g'(u)u'(x) \\ &= -u^{-2} \cdot (-\sin(x)) \\ &= -(\cos(x))^{-2} \cdot (-\sin(x)) \\ &= \frac{\sin(x)}{(\cos(x))^2} \\ &= \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos(x)} \\ &= \tan(x)\sec(x) \end{aligned}$$

2. $f(x) = \sqrt{x}\cos(x)$

   This is a product, with $u = \sqrt{x} = x^{1/2}$ and $v = \cos(x)$. Thus we have

$$\begin{aligned} f'(x) &= u'v + v'u \\ &= \frac{1}{2}x^{-1/2}\cos(x) + x^{1/2}(-\sin(x)) \\ &= \frac{\cos(x)}{2\sqrt{x}} - \sqrt{x}\sin(x) \end{aligned}$$

3. $f(x) = \dfrac{3\ln(x)}{e^x}$

This is a quotient, with $u = 3\ln(x)$ and $v = e^x$. Thus we have

$$\begin{aligned} f'(x) &= \frac{vu' - uv'}{v^2} \\ &= \frac{\frac{d}{dx}(3\ln(x)) \cdot e^x - 3\ln(x) \cdot \frac{d}{dx}(e^x)}{(e^x)^2} \\ &= \frac{\frac{3}{x} \cdot e^x - 3\ln(x)e^x}{(e^x)^2} \\ &= \frac{3e^x\left(\frac{1}{x} - \ln(x)\right)}{(e^x)^2} \\ &= \frac{3\left(\frac{1}{x} - \ln(x)\right)}{e^x} \end{aligned}$$

4. $f(x) = \dfrac{4}{(x^3 + 25x)^3}$

I could either do this using the quotient rule or just using the chain rule. I'll show you both ways, so you believe that you get the same answer either way!

- **Using the chain rule:**
  I can easily rewrite $f(x)$ as
  $$f(x) = 4(x^3 + 25x)^{-3}.$$
  When I go to differentiate it, the outer function will be $g(u) = 4u^{-3}$ and the inner function will be $u(x) = x^3 + 25x$.
  $$\begin{aligned} f'(x) &= g'(u)u'(x) \\ &= -12u^{-4}(3x^2 + 25) \\ &= -12(x^3 + 25x)^{-4}(3x^2 + 25) \\ &= -\frac{12(3x^2 + 25)}{(x^3 + 25x)^4} \end{aligned}$$

- **Using the quotient rule:**
  Here $u = 4$ (a constant! It's derivative will be 0, when the time comes) and $v = (x^3 + 25x)^3$ (a composition – I'll need the chain rule at some point, and when I do, my choices of $g(u)$ and u will be almost the same as if I'd done it only with the chain rule, as above.).
  Diving right into the quotient rule, we have

$$\begin{aligned} f'(x) &= \frac{vu' - uv'}{v^2} \\ &= \frac{(x^3 + 25x)^3 \cdot 0 - 4 \cdot 3(x^3 + 25x)^2(3x^2 + 25)}{((x^3 + 25x)^3)^2} \\ &= -\frac{4 \cdot 3(x^3 + 25x)^2(3x^2 + 25)}{(x^3 + 25x)^6} \\ &= -\frac{12(3x^2 + 25)}{(x^3 + 25x)^4} \end{aligned}$$

  I did indeed get the same thing either way. Using the quotient rule is not the easiest way to do this problem, since you end up having to use the chain rule anyway, but knowing that it works takes some pressure off you – even if you don't recognize that you could write this as just a power, you can still get the correct answer – *as long as you remember that the derivative of a constant is 0!*

5. $f(x) = \dfrac{x \ln(x) - x}{5}$

   First of all, notice that we can write this as $f(x) = \frac{1}{5}(x \ln(x) - x)$. Since $\frac{1}{5}$ is just acting as a multiplicative constant, there's no need to use the quotient rule on this problem. However, inside the parentheses, one of the terms is a product, so I'll have to use the product rule when it comes time to differentiate that term.

$$
\begin{aligned}
f'(x) &= \frac{1}{5}\left(\frac{d}{dx}(x)\ln(x) + x\frac{d}{dx}(\ln(x)) - 1\right) \\
&= \frac{1}{5}(\ln(x) + x \cdot (1) - 1) \\
&= \frac{1}{5}\ln(x)
\end{aligned}
$$

6. $f(x) = \dfrac{\cot(x)}{3e^x}$

   Remember: the derivative of $\cot(x)$ is $-(\csc(x))^2 = \sec^2(x)$.

$$
\begin{aligned}
f'(x) &= \frac{(3e^x)\frac{d}{dx}(\cot(x)) - \cot(x)\frac{d}{dx}(3e^x)}{(3e^x)^2} \\
&= \frac{3e^x\left(-\csc(x)\right)^2 - \cot(x)(3e^x)}{(3e^x)^2} \\
&= \frac{3e^x\left(-\csc^2(x) - \cot(x)\right)}{(3e^x)^2} \\
&= \frac{-\csc^2(x) - \cot(x)}{3e^x}
\end{aligned}
$$

7. $f(x) = e^x \csc(x)(x^2+1)$

   We have a triple product. We can approach these by just differentiating one part of the product at a time, and leaving the other parts untouched!

   Remember: $\dfrac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$.

   $u(x) = e^x$, $v(x) = \csc(x)$ and $w(x) = x^2 + 1$.

$$
\begin{aligned}
f'(x) &= u'vw + uv'w + uvw' \\
&= e^x\csc(x)(x^2+1) + e^x(-\csc(x)\cot(x))(x^2+1) + e^x\csc(x)(2x) \\
&= e^x[(x^2+1)\csc(x)(1-\cot(x)) + 2x\csc(x)]
\end{aligned}
$$

8. $f(x) = \dfrac{x\cos(x)}{\ln(x)}$

   Not to panic because we have a combination of a product and a quotient! Just take it piece by piece. The first thing we see is the quotient, with $u(x) = x\cos(x)$ and $v(x) = \ln(x)$. So we just see where that takes us:

$$f'(x) = \frac{\ln(x)\cdot\frac{d}{dx}(x\cos(x)) - x\cos(x)\cdot\frac{d}{dx}(\ln(x))}{(\ln(x))^2}.$$

   When we look at this, we see that we have to take the derivative of $x\cos(x)$ – but we know how to do that – just the product rule, no big deal.

$$\begin{aligned} f'(x) &= \frac{\ln(x)\cdot[\frac{d}{dx}(x)\cos(x) + x\cdot\frac{d}{dx}(\cos(x))] - x\cos(x)(\frac{1}{x})}{(\ln(x))^2} \\ &= \frac{\ln(x)\cdot[\cos(x) + x\cdot(-\sin(x))] - \cos(x)}{(\ln(x))^2} \\ &= \frac{\ln(x)\cos(x) - x\ln(x)\sin(x) - \cos(x)}{(\ln(x))^2} \end{aligned}$$

9. $f(x) = e^{x^3\sin(x)}$

   I've got $e^{\text{a function}}$, so I know I'm looking at the chain rule. So I dive right in: $f = g(u(x))$, where $g(u) = e^u$ and $u(x) = x^3\sin(x)$.

   (I realize as soon as I write down what u is that I'm going to have to use the product rule at some point, but I'll cross that bridge when I come to it!)

$$\begin{aligned} f'(x) &= g'(u)u'(x) \\ &= e^u\cdot\frac{d}{dx}(x^3\sin(x)) \text{ Here's where I need to use the product rule!)} \\ &= e^{x^3\sin(x)}[x^3\cos(x) + 3x^2\sin(x)] \end{aligned}$$

10. $f(x) = \dfrac{(x^3+5x)^{100}}{\ln(x)}$

Using the quotient rule, with $u = (x^3+5x)^{100}$ and $v = \ln(x)$, I find

$$f'(x) = \frac{\ln(x)\cdot\frac{d}{dx}\left((x^3+5x)^{100}\right) - (x^3+5x)^{100}\cdot\frac{1}{x}}{(\ln(x))^2}$$

I'm going to have to use the chain rule on $\frac{d}{dx}\left((x^3+5x)^{100}\right)$

The outer function is u^{100} and the inner function is $u(x) = x^3+5x$.

$$= \frac{\ln(x)\cdot 100(x^3+5x)^{99}(3x^2+5) - (x^3+5x)^{100}\cdot\frac{1}{x}}{(\ln(x))^2}$$

$$= \frac{(x^3+5x)^{99}\left(100\ln(x)(3x^2+5) - \frac{x^3+5x}{x}\right)}{(\ln(x))^2}$$

$$= \frac{(x^3+5x)^{99}(\ln(x)(300x^2+500) - x^2 - 5)}{(\ln(x))^2}$$

11. $f(x) = \sec(x^3)$ (You can now use what you found the derivative of $\sec(x)$ to be.)

This is clearly a composition, with $g(u) = \sec(u)$ and $u(x) = x^3$. What ends up making this tricky is the form the derivative of $\sec(u)$ takes. Remember, we found earlier that the derivative of $\sec(u) = \sec(u)\tan(u)$. Notice that the u must appear in both the secant *and* the tangent!

So:

$$\begin{aligned} f'(x) &= g'(u)u'(x) \\ &= \sec(u)\tan(u)(3x^2) \\ &= 3x^2\sec(x^3)\tan(x^3) \end{aligned}$$

12. $f(x) = \sqrt{\cos(x^4 - \frac{7}{x})}$

Again, this is clearly a composition. I can use the chain rule, with $g(u) = \sqrt{u}$ and $u(x) = \cos(x^4 - \frac{7}{x})$. But as soon as I write this, I realize that when it comes time to differentiate $u(x)$, I'll be using the chain rule again. I'll cross that bridge when I come to it, however.

$$\begin{aligned} f'(x) &= g'(u)u'(x) \\ &= \frac{1}{2}u^{-1/2} \cdot \frac{d}{dx}\left(\cos(x^4 - \frac{7}{x})\right) \end{aligned}$$

Remember $\sqrt{u} = u^{1/2}$, which is what gives us $\frac{1}{2}u^{-1/2}$ as its derivative.

Now I'm faced with differentiating $\cos(x^4 - \frac{7}{x})$.

In order to do that, I'll use $g(u) = \cos(u)$ and $u(x) = x^4 - \frac{7}{x} = x^4 - 7x^{-1}$

$$\begin{aligned} &= \frac{1}{2}\left(\cos(x^4 - \frac{7}{x})\right)^{-1/2} \cdot g'(u)u'(x) \\ &= \frac{1}{2}\left(\cos(x^4 - \frac{7}{x})\right)^{-1/2} \cdot -\sin(u)(4x^3 + 7x^{-2}) \\ &= -\frac{1}{2}\left(\cos(x^4 - \frac{7}{x})\right)^{-1/2} \cdot \sin(x^4 - \frac{7}{x})(4x^3 + 7x^{-2}) \end{aligned}$$

13. $f(x) = \ln\left(\frac{x^3}{e^x}\right)$

I see natural log with something that's not x in it, so I know I'm using the chain rule, with $g(u) = \ln(u)$ and $u(x) = \frac{x^3}{e^x}$. I also realize, as I'm writing down what u is, that when it comes time to differentiate $u(x)$, I'm going to have to use the chain rule.

$$\begin{aligned}
f'(x) &= g'(u)u'(x) \\
&= \frac{1}{u}u'(x) \\
&= \frac{1}{\frac{x^3}{e^x}} \cdot \frac{e^x(3x^2) - x^3(e^x)}{(e^x)^2} \\
&= \frac{e^x}{x^3} \cdot \frac{e^x(3x^2 - x^3)}{(e^x)^2} \\
&= \frac{x^2(3-x)}{x^3} \\
&= \frac{3-x}{x} \\
&= \frac{3}{x} - 1
\end{aligned}$$

14. $f(x) = \tan(x^4)\sec^4(x)$

    Using the product rule on this, I get that

$$\begin{aligned}
f'(x) &= \tan(x^4) \cdot \frac{d}{dx}((\sec(x))^4) + \sec^4(x) \cdot \frac{d}{dx}(\tan(x^4)) \\
&\quad \text{I'm going to have to use the chain rule on both of the derivatives left!} \\
&= \tan(x^4) \cdot 4(\sec(x))^3(\sec(x)\tan(x)) + \sec^4(x) \cdot \sec^2(x^4)(4x^3) \\
&= 4\tan(x^4)\sec^4(x)\tan(x) + 4x^3\sec^4(x)\sec^2(x^4) \\
&= 4\sec^4(x)\left(\tan(x^4)\tan(x) + x^3\sec^2(x^4)\right)
\end{aligned}$$

15. $f(x) = \cos\left(\frac{1}{(x^2-5x)^4}\right)$

    I can rewrite $f(x)$ as

$$f(x) = \cos((x^2 - 5x)^{-4}).$$

This gives me an inner function, a middle function, and an outer function, so I'm looking at doing the chain rule twice. No big deal, I'll just work from the outside in, and take it as it comes.

I'll begin with $g(u) = \cos(u)$ and $u(x) = (x^2 - 5x)^{-4}$.

$$\begin{aligned} f'(x) &= g'(u)u'(x) \\ &= -\sin(u)u'(x) \\ &= -\sin((x^2-5x)^{-4}) \cdot \frac{d}{dx}((x^2-5x)^{-4}) \\ &\quad \text{Now, I'll use } g(u) = u^{-4} \text{ and } u(x) = x^2 - 5x. \\ &= -\sin((x^2-5x)^{-4}) \cdot -4(x^2-5x)^{-5}(2x-5) \\ &= \frac{4(2x-5)\sin\left(\frac{1}{(x^2-5x)^4}\right)}{(x^2-5x)^5} \end{aligned}$$

16. $f(x) = \dfrac{\ln(x^7)}{\cos(e^x)}$

    This is a quotient, but both the top and the bottom are compositions. So I'm going to begin with the quotient rule, and then move right into the chain rule, when I need it.

$$\begin{aligned} f'(x) &= \frac{\cos(e^x) \cdot \frac{1}{x^7} \cdot 7x^6 - \ln(x^7) \cdot -\sin(e^x)e^x}{\cos^2(e^x)} \\ &= \frac{\frac{7\cos(e^x)}{x} + e^x \ln(x^7)\sin(e^x)}{\cos^2(e^x)} \end{aligned}$$

17. $f(x) = e^{\tan(\ln(x))}$

    Using $g(u) = e^u$ and $u(x) = \tan(\ln(x))$ in the chain rule, I find

$$\begin{aligned} f'(x) &= g'(u)u'(x) \\ &= e^u \cdot \frac{d}{dx}(\tan(\ln(x))) \\ &= e^{\tan(\ln(x))} \cdot \sec^2(\ln(x)) \cdot \frac{1}{x} \end{aligned}$$

18. $f(x) = x^{1/3}\sin(x)\sec(x)$

    Another triple product!

$$\begin{aligned} f'(x) &= x^{1/3}\sin(x)\sec(x)\tan(x) + x^{1/3}\cos(x)\sec(x) + \frac{1}{3}x^{-2/3}\sin(x)\sec(x) \\ &\quad \text{But } \sec(x) \text{ is } \frac{1}{\cos(x)} \text{ and } \tan(x) = \frac{\sin(x)}{\cos(x)}, \text{ so ...} \\ &= x^{1/3}(\tan(x))^2 + x^{1/3} + \frac{1}{3}x^{-2/3}\tan(x) \end{aligned}$$