1. Find the maximum and minimum value of $f(x) = x^3 - 3x + 5$ over the interval $[0, 2]$.

   In general, the maximum and minimum values of a function f can only *occur* at the following places:

   - stationary points of f (i.e. where $f'(x) = 0$)
   - places where the derivative does not exist
   - at the endpoints of the interval.

   Be aware: This doesn't mean each of these points *is* some sort of maximum or minimum. It just means that instead of testing every single x-value between $x = 0$ and $x = 2$, we can narrow the field down to points of the above three types.

   To find the stationary points and places where the derivative does not exist, we find the derivative:

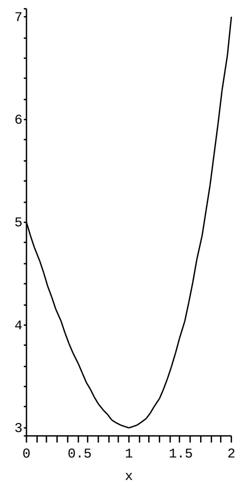
   $$f'(x) = 3x^2-3 = 3(x^2-1) = 3(x-1)(x+1) \qquad f'(x) = 0 \text{ at } x = 1, x = -1.$$

   Since $x = -1$ isn't in the interval $[0, 2]$, the only places f can attain its maximum and minimum value are therefore $x = 0$, $x = 1$, and $x = 2$.

x	$f(x)$
0	5
1	3
2	7

   Therefore, without even looking at the graph of $f(x)$, I know that the highest point of f between 0 and 2 occurs at $x = 2$ and the lowest occurs at $x = 1$.

   To check this, I can of course look at the graph.



Sure enough, the highest point occurs not at a flat part (local max) but at the endpoint $x = 2$, while the lowest point occurs at the local minimum at $x = 1$.

2. Find a function with stationary points at $x = -1$, $x = 0$ and $x = 2$. Check that your function behaves properly by graphing it.

   Stationary points are where the derivative is 0. So we want to construct a function f that has $f'(-1) = 0$, $f'(0) = 0$, and $f'(2) = 0$. How can I create such a function? There are probably an infinite number of such functions, but here's what I think is the easiest way:

   What we're going to do is create f' first, and then get f from there.

   - Could $f'(x) = 0$–that is, could $f'(x)$ be the horizontal line that is the x-axis? Well, if $f'(x) = 0$, then $f'(-1) = 0$, $f'(0) = 0$ and $f'(2) = 0$, but these aren't the *only* points where f' is 0, so we wouldn't have created a function with only these stationary points.
   - The next-easiest function f' that has $f'(0) = 0$ is $f'(x) = x$. Unfortunately, if $f'(x) = x$, then $f'(-1) \neq 0$ and $f'(2) \neq 0$. But we can build on this idea!

- Along the same ideas as the previous thought, $f'(x) = x + 1$ has $f'(-1) = 0$, but $f'(0) \neq 0$. *However* (and here' s the big brilliant idea!), the function $x(x + 1)$ *is* 0 at both $x = 0$ and $x = -1$. Building upon that idea even farther: since $x - 2$ is 0 at $x = 2$, if we throw that factor into the mix as well, what do we get?

  $f'(x) = x(x+1)(x-2) \qquad f'(0) = 0 \qquad f'(-1) = 0 \qquad f'(2) = 0.$

  Thus $f'(x) = x(x + 1)(x - 2)$ will work as our derivative.

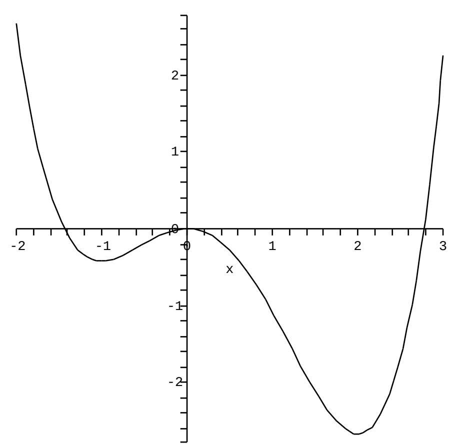
In that case, what is $f(x)$? We'll have to antidifferentiate $f'(x)$. In order to do that, we first must simplify $f'(x)$:

$f'(x) = x(x+1)(x-2) = x[(x+1)(x-2)] = x(x^2-2x+x-2) = x(x^2-x-2) = x^3-x^2-2x.$

Antidifferentiating this, we find that one function with stationary points at $x = -1$, $x = 0$, and $x = 2$ would be

$$f(x) = \frac{x^4}{4} - \frac{x^3}{3} - x^2.$$

We can always check our answer, either by differentiating it and making sure we get stationary points at $x = 0$, $x = -1$ and $x = 2$ or simply by graphing our $f(x)$:



Sure enough, this has stationary points at $x = -1$, $x = 0$ and $x = 2$ just as desired.

3. For which values of k, if any, does the function $f(x) = (8x+k)/x^2$ have a local minimum at $x = 4$?

   $f(x)$ will have a local minimum at $x = 4$ if

   - $f'(4) = 0$, and
   - $f''(4) > 0$ (since then the function would be concave up there).

   - First, let's see what I can tell about k just from knowing $f'(4) = 0$ – that is, just from knowing that f has a stationary point at $x = 4$.

     I need to $f'(4)$ to be 0. So I need to find $f'(x)$, plug 4 in, set the result equal to 0, and see what that tells me about k. But right now, $f(x)$ isn't in a form we know how to differentiate, so I need to simplify/rewrite f first.

$$\begin{aligned} f(x) &= \frac{8x+k}{x^2} \\ &= \frac{8x}{x^2} + \frac{k}{x^2} \\ &= \frac{8}{x} + \frac{k}{x^2} \text{ if } x \neq 0 \\ &= 8x^{-1} + kx^{-2} \end{aligned}$$

Now f is in a form I can differentiate.

$$\begin{aligned} f'(x) &= -8x^{-2} - 2kx^{-3} \\ &= -\frac{8}{x^2} - \frac{2k}{x^3} \end{aligned}$$

I know that in order for $x = 4$ to be a local minimum, I need $f'(4) = 0$. Let's see where that takes us:

$$\begin{aligned} f'(4) &= -\frac{8}{16} - \frac{2k}{64} \\ &= -\frac{1}{2} - \frac{k}{32} \end{aligned}$$

Since $f'(4) = 0$, this means that

$$-\frac{1}{2} - \frac{k}{32} = 0 \Rightarrow \frac{k}{32} = -\frac{1}{2} \Rightarrow k = -16.$$

Conclusion: In order for f to have any sort of stationary point at $x = 4$, k must be -16. But we don't know yet whether f has a local minimum, local maximum, or inflection point at $x = 4$ when $k = -16$, so even though it feels as if we're done, we're not!

- If we let $k = -16$, is $f''(4) > 0$?
  With $k = -16$, we have that

$$\begin{aligned} f(x) &= (8x - 16)/x^2 \\ &= 8x^{-1} - 16x^{-2} \end{aligned}$$

$$\Rightarrow f'(x) = -8x^{-2} + 32x^{-3}$$
$$\Rightarrow f''(x) = -16x^{-3} - 96x^{-4}$$

$$\begin{aligned} \Rightarrow f''(4) &= \frac{16}{4^3} - \frac{96}{4^4} \\ &= \frac{1}{4} - \frac{2^5 \cdot 3}{2^8} \\ &= \frac{1}{4} - \frac{3}{8} \\ &< 0 \end{aligned}$$

  Because $f''(4)$ is negative at $x = 4$, f is concave down at $x = 4$, which means that f has a *maximum* at $x = 4$, not a *minimum* after all.

Conclusion: There is no value of k for which $x = 4$ is a local minimum of $f(x)$, although $k = -16$ does lead to $x = 4$ being a local maximum.

4. A particle moves along a straight line so that its velocity after t minutes is given by $v(t) = t^2$. How far does the particle travel between $t = 1$ and $t = 3$?

   If its velocity is given by $v(t) = t^2$, then its position must be given $p(t) = t^3/3$. Thus at $t = 1$, it was at position $p(1) = 1/3$ and at $t = 3$ it was at position $p(3) = 27/3 = 9$. At no time in that interval did it turn around, since the velocity was always positive, and so the difference in position is the same as the distance traveled.

   Thus it traveled 26/3 units in those 2 minutes.