1. **Radioactive Decay** Living plants and animals contain carbon-14 in known proportions. When an organism dies, carbon-14 decays radioactively at a known rate. By measuring the proportion of carbon-14 present in a sample of organic matter (such as long-dead wood), scientists can estimate the sample's age. This method is called **carbon dating**.

   More specifically, carbon-14 decays at a rate proportional to the mass of ^{14}C present, and the half-life of ^{14}C is 5,730 years. That is, at the end of any 5,730 year period, half the atoms that were radioactive at the beginning of the period are still radioactive, and half have decayed into non-radioactive atoms (specifically, into nitrogen-14) .

   (a) Find the *decay constant* k for ^{14}C.

   Hints:

   - What do you know, because the rate of decay is proportional to the amount present?
   - If one-half have decayed, half are left.
   - You don't have to know the initial amount of ^{14}C present – just call it A_0.

   Because the rate of decay is proportional to the amount present at time t, we know that

   $$A' = kA.$$

   We know that the solution to this differential equation is

   $$A(t) = A(0)e^{kt}.$$

   We also know that the half-life of ^{14}C is $5,730$ years, or in other words, after $5,730$ years, half of what you started with will be left. Mathematically, we could write this as

   $$A(5730) = \frac{1}{2}A_0.$$

Putting this all together, we can solve for the decay constant k:

$$\begin{aligned}
\frac{1}{2}A_0 = A(5730) &= A_0 e^{k\cdot 5730} \\
\frac{1}{2} &= e^{5730k} \\
\ln\left(\frac{1}{2}\right) &= 5730k \\
k &= \frac{\ln(\frac{1}{2})}{5730} \approx \frac{-0.6931471806}{5730} \approx -0.000121
\end{aligned}$$

So the decay constant is roughly -0.000121, and so when dealing with any radioactive carbon problem, we can use the formula

$$A(t) = A_0 e^{-0.000121t}.$$

(b) (Section 2.6 #36) How long does it take for one-quarter of the ^{14}C atoms in a sample to decay?

Hints:

- Use the decay constant you found in (a).
- Because the decay constant depends on t being measured in years, your answer will be in years.
- If one-quarter have decayed, three-quarters are left
- Since you don't know the mass of the ^{14}C initially present in the sample, just let A_0 represent $A(0)$.

Let $A(t)$ represent the mass of ^{14}C present t years after the instant of death.

As we found in part (a), the amount of ^{14}C t years after death will be given by

$$A_0 e^{-0.000121t}.$$

We want to know how long it takes for one-quarter of the ^{14}C atoms to decay, or in other words, how long it takes for there to be exactly three-quarters of the ^{14} atoms to be left. In other words, we want to know for what t $A(t) = \frac{3}{4}A_0$.

$$\begin{aligned}
\frac{3}{4}A_0 &= A_0 e^{-0.000121t} \\
\frac{3}{4} &= e^{-0.000121t} \\
\ln\left(\frac{3}{4}\right) &= -0.000121t \\
t &= \frac{\ln(\frac{3}{4})}{-0.000121} \approx \frac{-0.287682}{-0.000121} \approx 2377.5
\end{aligned}$$

Thus it takes roughly 2,377.5 years for one-quarter of the atoms in a sample to decay.

(c) (Section 26 #67) Human skeletal fragments were brought to a laboratory for carbon dating. Analysis showed that the proportion of ${}^{14}C$ to ${}^{12}C$ was only 6.25% of the value in living tissue. How long ago did this person die?

Hint: Let $t = 0$ correspond with the instant of death, and measure t in years.

First of all, after the previous problem, we know the person died a long, long, long time ago.

We'll assume that we continue to use the relationship $A(t) = A_0 e^{-0.000121t}$. Since ${}^{14}C$ decays into nitrogen, it shouldn't affect the amount of ${}^{12}C$, and so this should make sense. Thus the proportion of ${}^{14}C$ to ${}^{12}C$ should depend only on how much ${}^{14}C$ is present, and so we can continue to use

$$A(t) = A_0 e^{-0.000121t}.$$

If $t = 0$ corresponds to the time of death, we want to know how much time has passed when the amount of ${}^{14}C$ present is $.0625A(0) = .0625A_0$.

$$\begin{aligned} .0625A_0 &= A_0e^{-0.000121t} \\ .0625 &= e^{-0.000121t} \\ \ln(.0625) &= -0.000121t \\ t &= \frac{\ln(.0625}{-0.000121} \approx \frac{-2.772589}{-0.000121} \approx 22,914 \end{aligned}$$

Thus the person died roughly 22,914 years ago.

2. (Section 2.5 #42) A flu epidemic spreads through a 3000-student college community at a rate proportional to the product of the number of members already infected and the number of those not yet infected. (This product measures the number of possible infectious contacts.) Let $P(t)$ represent the number of students infected after t days. Write a differential equation that relates P and P'.

   Let $P(t)$ represent the number of students infected after t days.

   - The rate the flu spreads is P'.
   - The number of students infected is P, and the number of those not yet infected is $3000 - P$ (counting everyone either as infected or not-yet infected).

   Thus if the rate is proportional to the product of those infected and those not-yet infected, we translate this into

$$P' = k \cdot P(3000 - P),$$

   where k is the constant of proportionality.

3. (Section 2.6 #64) A mold grows at a rate proportional to the amount present. Initially, its weight is 2g; after 2 days, it weighs 5 g. How much does it weigh after 8 days?

   - Let $W(t)$ represent the weight of the mold after t days.

- Since the mold grows at a rate proportional to the amount present, $W' = kW$, and so $W(t) = W(0)e^{kt}$ for some growth constant k.
- $W(0) = 2$g, so $W(t) = 2e^{kt}$.
- $W(2) = 5$g
- We want to know $W(8)$.

We first need to find the growth constant k.

$$\begin{aligned} 5 = W(2) &= 2e^{k\cdot 2} \\ \frac{5}{2} &= e^{2k} \\ \ln\left(\frac{5}{2}\right) &= 2k \\ k &= \frac{\ln(5/2)}{2} \end{aligned}$$

Thus the weight of this mold after t days is

$$W(t) = 2e^{\frac{\ln(5/2)}{2}\cdot t} = 2e^{t\ln(5/2)/2},$$

and so after 8 days, the weight of this mold will be

$$W(8) = 2e^{8\ln(5/2)/2} = 2e^{4\ln(5/2)} \approx 78.125.$$

so after 8 days, the mold will way 78.125g.