

1. Evaluate (or approximate) the following integrals.

$$(a) \int_0^3 4e^x x + 2e^x x^2 dx$$

If $f(x) = 4e^x x + 2e^x x^2$, then I know from the FTC that

$$\int_0^3 4e^x x + 2e^x x^2 dx = F(3) - F(0),$$

where $F(x)$ is any antiderivative of $4e^x x + 2e^x x^2$.

So...

- **Goal 1:** find an antiderivative $F(x)$ of $4e^x x + 2e^x x^2$.
 - Both $4e^x x$ and $2e^x x^2$ are products. In differentiation, we have seen products come from the chain rule and the product rule.
 - Since the product rule produces a sum (which we have here), look to see whether the factors of the first product are related to the factors of the second by differentiation.
 - The sum that comes from the product rule has the form $(uv)' = uv' + u'v$. One of the factors in each product is e^x (which could be both u or u'). The remaining factors in the two products are $2x^2$ and $4x$. Sure enough, these are related to each other by differentiation. If we let $v = 2x^2$, then v' must be $4x$, and we have that

$$f(x) = uv' + u'v,$$

so an antiderivative should be

$$F(x) = uv = 2x^2 e^x.$$

- *Check:* $F'(x) = 2x^2 e^x + 4x e^x$, which is what we started with!
- *Conclusion:* $F(x) = 2x^2 e^x$.
- **Goal 2:** find the value of the definite integral

Using FTC v2,

$$\int_0^3 4e^x x + 2e^x x^2 dx = F(3) - F(0) = [2(3)^2 e^3] - [2(0)^2 e^0] = 2(9)e^3 - 0 = 18e^3.$$

$$(b) \int_0^1 x^{12} e^{x^{13}} dx$$

If $f(x) = x^{12}e^{x^{13}}$, then I know from the FTC that

$$\int_0^1 x^{12} e^{x^{13}} dx = F(1) - F(0),$$

where $F(x)$ is any antiderivative of $x^{12}e^{x^{13}}$.

So ...

- **Goal 1:** find an antiderivative $F(x)$ of $x^{12}e^{x^{13}}$.
 - $x^{12}e^{x^{13}}$ is a product. In differentiation, we have seen products come from the chain rule and the product rule.
 - F is unlikely to itself be a product, since the product rule produces a sum, and here we only have a single term.
 - F is most likely a composition!
 - When I look more closely, I see that there is indeed already a composition present: $e^{x^{13}}$.
 - I know that $\frac{d}{dx}(x^{13})$ is $13x^{12}$ – which would account for the factor of x^{12} I have.
 - The product that comes from the chain rule is

$$[f(u)]' = f'(u)u'.$$

The composition I have left is $e^{x^{13}}$. The "inside" of that is $u = x^{13}$. $u' = 13x^{12}$.

- *Try:* $F(x) = e^{x^{13}}$.
- *Check:* $F'(x) = e^{x^{13}} \cdot 13x^{12}$.

This is not quite what I'm integrating. What we have here is 13 times what we want.

- *Try:* Dividing what we tried before by 13:

$$F(x) = \frac{1}{13} e^{x^{13}}.$$

- *Check:* $F'(x) = \frac{1}{13} e^{x^{13}} \cdot 13x^{12} = x^{12}e^{x^{13}}$
- **Conclusion:** $F(x) = \frac{1}{13} e^{x^{13}}$

- **Goal 2:** find the value of the definite integral
Using FTC v2,

$$\int_0^1 x^{12} e^{x^{13}} dx = F(1) - F(0) = \left[\frac{1}{13} e^{x^{13}}\right] - \left[\frac{1}{13} e^{0^{13}}\right] = \frac{1}{13}(e - 1).$$

(c) $\int_{-1}^1 \sqrt{1 - x^2} dx$

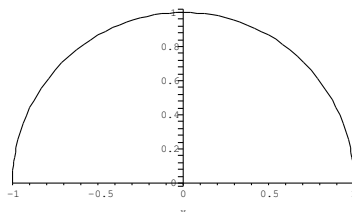
If $f(x) = \sqrt{1 - x^2} dx$, then I know from the FTC that

$$\int_{-1}^1 \sqrt{1 - x^2} dx = F(1) - F(-1),$$

where $F(x)$ is any antiderivative of $\sqrt{1 - x^2}$.

So ...

- **Goal 1:** find an antiderivative $F(x)$ of $\sqrt{1 - x^2}$.
 - Whatever F might be, it needs to differentiate into the above composition.
 - That suggests that my original function needs to also be a composition.
 - When I use the chain rule to differentiate a composition, the inside stays the same, and then I multiply by the derivative of the inside.
 - My *integrand* is not a product.
 - Uh oh! We're not going to be able to find an antiderivative anytime soon, if ever!
- **New Goal 1:** think about this differently!
The FTC, v2, isn't going to help us this time. But because $\sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$, we know from the FTC, v1, that this signed area exists – we can also just look at the graph to see that the signed area exists. So ... let's look at the graph and see if it's helpful!



Aha! The value of the definite integral is (by *definition* of the integral) simply the signed area between the curve and the x -axis from -1 to 1 . Looking at this, we can see that this is just the area of a semi-circle of radius 1 .

Therefore,

$$\int_{-1}^1 \sqrt{1-x^2} dx = \text{half the area of a circle of radius } 1 = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}.$$

2. Find the area of the region bounded by the graphs $y = x^2$ and $y = 2x + 3$.

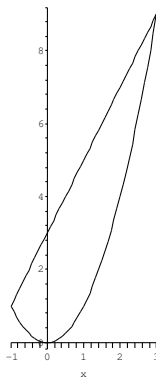
Hmmm. We haven't done anything quite like this before – we'd better look at a picture!

I know that $y = x^2$ is a parabola, and $y = 2x + 3$ is a line, but since I didn't give you any limits of integration, you need to find where the two curves intersect.

So, we set the two curves equal to each other:

$$\begin{aligned}x^2 &= 2x + 3 \\x^2 - 2x - 3 &= 0 \\(x - 3)(x + 1) &= \\x = 3 \text{ or } x = -1\end{aligned}$$

So the two graphs plotted together look like:



We know that the integral gives the signed area between a curve and the x -axis. But we only want the area below the line $y = 2x + 3$ and above the parabola. How can we get that?

Well...in essence, what we just said is we want to take the area below the line, and from that, throw away the area below the parabola! In

other words,

$$\begin{aligned}
 A &= \text{signed area between } 2x + 3 \text{ and } x \text{ axis} - \text{signed area between } x^2 \text{ and } x\text{-axis} \\
 &= \int_{-1}^3 2x + 3 \, dx - \int_{-1}^3 3x^2 \, dx \\
 &= \int_{-1}^3 2x + 3 - x^2 \, dx \\
 &= [x^2 + 3x - \frac{1}{3}x^3] \text{ from } -1 \text{ to } 3 \\
 &= [(3)^2 + 3(3) - \frac{1}{3}(3)^3] - [(-1)^2 + 3(-1) - \frac{1}{3}(-1)^3] \\
 &= [9 + 9 - 9] - [1 - 3 - \frac{1}{3}(-1)] \\
 &= 9 - [-2 + \frac{1}{3}] \\
 &= 9 - (-5/3) \\
 &= 32/3
 \end{aligned}$$

3. Let $f(t) = 2t \cos(t^2)$ and $F(x) = \int_1^x f(t) \, dt$.

(a) Find the equation of the line tangent to $y = F(x)$ at $x = 3$.

To find the equation of a tangent line, I need a point on the line and the slope of the line.

• **Point on the tangent line:**

The one point on the line that I have a hope of finding is the point of tangency: $(3, F(3))$.

Just plugging 3 in to the definition of $F(x)$, I find:

$$F(3) = \int_1^3 f(t) \, dt = \int_1^3 2t \cos(t^2) \, dt.$$

Once again, I need to antidifferentiate. As with 1(a) above, I end up with just a product, which means my antiderivative is *not* likely to itself be a product (since products differentiate to sums of products) – instead, it's most likely to be a composition (since the compositions differentiate to products). I look

more closely to see if there's a composition in my integrand, and I see that indeed there is $-\cos(t^2)$.

So, I'm looking for a simple composition that would differentiate to what I have. Insides stay the same, when differentiating a composition, so I know I have something with t^2 on the inside. I also know that $\sin(?)$ differentiates to $\cos(?)$, so

...

Try: antiderivative = $\sin(t^2)$.

Check: $\frac{d}{dt}(\sin(t^2)) = \cos(t^2) \cdot 2t$, which is what I have!

Therefore the FTC, v2 tells me that

$$F(3) = \sin(t^2) \text{ from } 1 \text{ to } 3 = \sin(9) - \sin(1) \approx -.43.$$

- **Slope of the tangent line:**

I know that the slope of the tangent line is the value of the derivative of F at the point of tangency: $F'(3)$.

First, I need to find $F'(x)$. Here, FTC, v1, comes to the rescue! We know that

$$\frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x),$$

so $F'(x) = f(x) = 2x \cos(x^2)$, and thus

$$F'(3) = 2(3) \cos(9) = 6 \cos(9) \approx -5.47.$$

- **Equation of the tangent line:**

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - (\sin(9) - \sin(1)) &= 6 \cos(9)(x - 3) \\ y + .43 &\approx -5.47(x - 3) \end{aligned}$$

(b) Find a formula for $\frac{d}{dx} (F(x^3))$.

Let $G(x) = F(x^3) = F(u)$, where $u(x) = x^3$.

Then the chain rule tells us that

$$G'(x) = F'(u)u'(x) = F'(u) \cdot 3x^2.$$

We know from the FTC, v1, that $F'(x) = f(x) = 2x \cos(x^2)$, so $F'(u) = 2u \cos(u^2) = 2x^3 \cos(x^6)$.

Therefore

$$\frac{d}{dx}(F(x^3)) = [2x^3 \cos(x^6)] \cdot (3x^2) = 6x^5 \cos(x^6).$$

In this particular case, because we could actually antidifferentiate $2x \cos(x^2)$, we could have done this another way, but I wanted to demonstrate this way, as it works even when we can't antidifferentiate the integrand.